# Black string entropy and Fourier-Mukai transform

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ABSTRACT: We propose a microscopic description of black strings in F-theory based on string duality and Fourier-Mukai transform. These strings admit several different microscopic descriptions involving D-brane as well as M2 or M5-brane configurations on elliptically fibered Calabi-Yau threefolds. In particular our results can also be interpreted as an asymptotic microstate count for D6-D2-D0 configurations in the limit of large D2-charge on the elliptic fiber. The leading behavior of the microstate degeneracy in this limit is shown to agree with the macroscopic entropy formula derived from the black string supergravity solution.

KEYWORDS: D-branes, Black Holes in String Theory.

# Contents

## 1. Introduction

Black hole microstate counting has been a problem of constant interest in string theory [1 – 9] for the past decade. This problem has been the subject of intense recent activity [10 – 27] motivated by the connection with topological strings proposed in [28] and by the correspondence between 4D black holes and 5D black holes [29] and black rings [30 – 33].

In $N = 2$ string theory compactifications, supersymmetric black holes can be described in terms of D-branes wrapping supersymmetric cycles in the internal manifold. The black hole entropy is determined by the degeneracy of D-brane bound states with fixed topological charges. In the semiclassical approximation, D-brane bound states are associated to cohomology classes on the moduli space of classical supersymmetric configurations. The macroscopic entropy formula is typically captured by the asymptotic growth of BPS degeneracies in the limit of large charges. This has been shown in [5, 6, 19] for D4-D2-D0 configurations on Calabi-Yau threefolds. Analogous results for D-brane configurations with nonzero D6-brane charge seem to be more elusive.

In this paper we address the problem of counting the microstate degeneracy for D-brane configurations with nonzero D6-brane charge on elliptically fibered Calabi-Yau threefolds. A string duality chain described in section two shows that this system admits several different descriptions in terms of wrapped branes in F-theory, M-theory or IIA compactifications. In particular this duality chain predicts an equivalence of the D6-D2-D0 system with a D4-D2-D0 configuration on the same Calabi-Yau threefold, which can be recognized as a Fourier-Mukai transform along the elliptic fibers. This is discussed in detail in section three. Another incarnation of the D6-D2-D0 configuration which will play an important

role in this paper is a noncritical six-dimensional string obtained by wrapping D3-branes on holomorphic curves in F-theory compactifications.

D-brane systems with D4-D2-D0 charges have a known microscopic CFT description [5, 6, 19] which allows one to compute the asymptotic degeneracy of states in the limit of large D0 charge. Our goal is to compare the resulting entropy formula with a macroscopic computation performed in a low energy supergravity description. We will show in section four that a reliable macroscopic description in the limit of large D0 charge must be formulated in terms of black-string solutions of $N = 1$ six dimensional supergravity. The resulting macroscopic entropy formula reproduces the macroscopic result including certain subleading corrections.

The problem of microstate degeneracies for D6-D4-D2-D0 black holes is also addressed in the upcoming work [34] using split attractor flows. Although this seems to be a different approach than the Fourier-Mukai transform employed here, it would be interesting to understand the relation between these two methods.

## Acknowledgments

**Note added.** This paper has some partial overlap with [35] which appeared when this work was close to completion.

## 2. Strings, D-branes and Fourier-Mukai transform

In this section we explain the connection between Fourier-Mukai transform, D-branes and noncritical strings in F-theory.
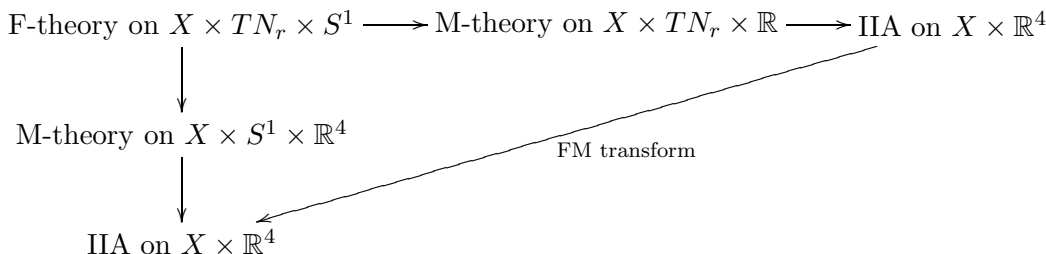
We will be interested in D6-D2-D0 configurations in IIA compactifications on an elliptic fibration $X$. Such configurations can be mapped to D4-D2-D0 configurations by a T-duality transformation on the elliptic fiber. From a mathematical point of view this a Fourier-Mukai transform mapping D-branes on $X$ to D-branes on the dual Calabi-Yau threefold $X^\vee$. Note that $X$ and $X^\vee$ are canonically isomorphic if $X$ is a smooth elliptic fibration with a section, which will be assumed from this point on. Therefore we will not make a distinction between $X$ and $X^\vee$ in the following.

The presence of degenerate elliptic fiber makes the T-duality transformation rather subtle since strictly speaking elliptic Calabi-Yau threefolds with $SU(3)$ holonomy do not admit isometries. For this reason it may be useful to note that this transformation can be physically understood as a sequence of string duality transformations relating noncritical strings in six dimensional F-theory compactifications to IIA D-brane configurations. This observation will also be helpful in identifying the correct macroscopic description of the system.

## 2.1 A duality chain

The basic observation due to [6] is that D3-branes wrapping a curve $C \subset B$ in the base of an F-theory compactification are related by string duality to either M2 or M5 brane configurations. This can be seen by mapping compactifying F-theory on an elliptic Calabi-Yau manifold $X$ times a circle $S^1$ of radius $R$. A D3-brane wrapping $C \times S^1$ equivalent to an M2-brane wrapping the curve $C$ while a D3-brane wrapping the curve $C$ but not $S^1$ is mapped to an M5-brane supported on the vertical divisor $D = \pi^{-1}(C)$ in $X$. Note that in both cases, we can obtain a IIA description of the system by further compactifying on an extra circle.

Motivated by the 4D/5D black hole correspondence, we will consider an F-theory background of the form $X \times TN_r \times S^1 \times \mathbb{R}$ where $TN_r$ is a Taub-NUT space of type $A_{r-1}$, and $\mathbb{R}$ is the time direction. This theory contains six-dimensional noncritical strings obtained as above by wrapping a D3-brane on $C \times S^1$, where $C$ is a curve in $B$. We perform again the duality transformations described in the above paragraph taking into account the presence of the Taub-NUT space. Namely we can reduce this system to an M-theory compactification by making either the circle $S^1$ or the circle fiber of the Taub-NUT space very small. In the first case the D3-brane will be mapped to a M2-configuration in M-theory on $X \times TN_r$, while in the second case we will obtain an M5-brane configuration on $X \times S^1 \times \mathbb{R}^4$. The second transformation involves the known NS five-brane Taub-NUT duality [43 – 45]. Finally, in both cases we can compactify the theory to a IIA model, obtaining different D-brane configurations. Namely the first sequence of transformations yields a D4-D2-D0 configuration while the second results in a D6-D2-D0 configuration. These configurations turn out to be related by a Fourier-Mukai transform. This paragraph can be summarized in the following diagram

F-theory on $X \times TN_r \times S^1 \longrightarrow$ M-theory on $X \times TN_r \times \mathbb{R} \longrightarrow$ IIA on $X \times \mathbb{R}^4$

M-theory on $X \times S^1 \times \mathbb{R}^4$

FM transform

IIA on $X \times \mathbb{R}^4$

Next we compute the effect of the Fourier-Mukai transform on D-brane charges, and show that it is in agreement with the above duality transformations up to curvature corrections which cannot be computed from physical considerations.

## 2.2 D-branes and Fourier-Mukai transform

Our set-up is a IIA compactification on a smooth elliptically fibered Calabi-Yau threefold $X$. In order to set the ground for our discussion, we will start with a short review of special Kähler geometry, BPS states and D-branes. Throughout this paper we will identify the complexified Kähler moduli space of $X$ with the complex structure moduli space of the mirror threefold $Y$. Let

$$\Pi = \left[\mathcal{F}_0, \mathcal{F}_A, X^A, X^0\right]^{\text{tr}} \tag{2.1}$$

denote the periods of the holomorphic three-form on $Y$, where $A = 1, \ldots, h^{1,1}(X)$. The inhomogeneous flat coordinates on the moduli space are

$$t^A = \frac{X^A}{X^0}, \qquad A = 1, \ldots, h^{1,1}(X). \tag{2.2}$$

The large radius limit point in the Kähler moduli space of $X$ is identified with a large complex structure (LCS) limit of $Y$. The periods are normalized so that $X^0$ is the fundamental period and $X^A$, are the logarithmic periods at the LCS point.

The central charge of a BPS state with charges $(P^\Lambda, Q_\Lambda)$, $\Lambda = 0, \ldots, h^{1,1}(X)$, is given by

$$\mathcal{Z} = e^{K/2}(Q_\Lambda X^\Lambda - P^\Lambda \mathcal{F}_\Lambda) \tag{2.3}$$

where

$$K = -\ln i(\overline{X}^\Lambda \mathcal{F}_\Lambda - X^\Lambda \overline{\mathcal{F}}_\Lambda) \tag{2.4}$$

is the Kähler potential.

From a microscopic point of view, BPS states are bound states of D6-D4-D2-D0 brane configurations on $X$. Such configurations are described by holomorphic vector bundles, or, more generally, coherent on sheaves on $X$. Given such an object $\mathcal{E}$, the central charge of the corresponding BPS state has an expansion of the form

$$\mathcal{Z} = e^{K/2} X^0 \left( \int_X e^{J(t^A)} \mathrm{ch}(\mathcal{E}) \sqrt{\mathrm{Td}(X)} + \cdots \right) \tag{2.5}$$

near the large radius point, where $J(t^A)$ denotes the complexified Kähler form on $X$, and ... stand for world-sheet one-loop and instanton corrections. Following the standard conventions in the literature we will use the notation

$$Z(\mathcal{E}) = \int_X e^{J(t^A)} \mathrm{ch}(\mathcal{E}) \sqrt{\mathrm{Td}(X)}. \tag{2.6}$$

More generally, if $\alpha$ is a cohomology class on $X$, we will denote by

$$Z(\alpha) = \int_X e^{J(t^A)} \alpha \sqrt{\mathrm{Td}(X)} \tag{2.7}$$

Mirror symmetry implies that the logarithmic periods $X^A$ have an expansion of the form

$$X^A = X^0(Z(\beta^A) + \cdots) \tag{2.8}$$

near the large radius limit, where $\beta^A \in H^{2,2}(X)$ are Poincaré dual to some curve classes $C_A$ on $X$. The remaining periods have similar expansions

$$\begin{aligned} \mathcal{F}_A &= X^0(Z(\alpha_A) + \ldots) \\ \mathcal{F}_0 &= X^0(Z(\mathcal{O}_X) + \ldots) \end{aligned} \tag{2.9}$$

where $\alpha_A \in H^{1,1}(X)$ is a basis of $H^{1,1}(X)$, and $\{\beta^A\}$ is a basis of $H^{2,2}(X)$ so that

$$\int_X \alpha_A \wedge \beta^B = \delta_A^B. \tag{2.10}$$

We can make a more specific choice of even cohomology generators taking into account the elliptic fibration structure of $X$. We will restrict ourselves to smooth elliptic fibrations $\pi : X \to B$ which can be written in Weierstrass form. The base $B$ is a smooth del Pezzo surface. Then $h^{1,1}(X) = h^{1,1}(B) + 1$ and we can choose the basis $\{\alpha_A\} \subset H^{1,1}(X)$ so that

$$\alpha_i = \pi^* \gamma_i, \qquad i = 1, \ldots h^{1,1}(B). \tag{2.11}$$

Moreover, the last basis element $\alpha_h$, where $h = h^{1,1}(X)$, is normalized so that

$$\int_F \alpha_h = 1, \qquad \int_C \alpha_h = 0,$$

where $F$ denotes the class of the elliptic fiber, and $C$ is an arbitrary horizontal curve class[1] on $X$. Denoting by $\sigma$ the $(1,1)$ class related by Poincaré duality to the section class, we have

$$\alpha_h = \sigma + \pi^* c_1(B). \tag{2.12}$$

Let $\{\eta^i\}$, $i = 1, \ldots, h^{1,1}(B)$ denote the dual basis of $H^{1,1}(B)$, i.e.

$$\int_B \gamma_i \wedge \eta^j = \delta_i^j. \tag{2.13}$$

Then we can choose the basis $\{\beta^A\} \subset H^{2,2}(X)$ so that

$$\beta^i = \sigma \wedge \pi^* \eta^i, \qquad i = 1, \ldots, h^{1,1}(B) \tag{2.14}$$

and $\beta^h$ is Poincaré dual to the fiber class $F$.

The D6-D2-D0 configurations related by duality to F-theory noncritical strings are described by holomorphic bundles $\mathcal{E}$ on $X$ with Chern character

$$\mathrm{ch}(\mathcal{E}) = r - \sum_{i=1}^{h^{1,1}(B)} q_i \beta^i - n \beta^h - m\omega \in H^0(X) \oplus H^{2,2}(X) \oplus H^{3,3}(X) \tag{2.15}$$

where $\omega \in H^{3,3}(X)$ is the fundamental class of $X$ normalized so that

$$\int_X \omega = 1.$$

A straightforward computation shows that

$$Z(\mathcal{E}) = rZ(\mathcal{O}_X) - q_i Z(\beta^i) - nZ(\beta^h) - m \tag{2.16}$$

This expression determines the charge vector of the corresponding BPS state

$$(P^0, P^A, Q_A, Q_0) = (r, 0, -q_i, -n, -m). \tag{2.17}$$

---

[1]A curve class will be called horizontal if it lies in the image of the pushforward map $\iota_* : H_2(B) \to H_2(X)$, where $\iota : B \to X$ is the canonical section of the Weierstrass model.

In the duality chain described in section (2.1), $r$ can be easily identified with the charge of the Taub-NUT space, $C = q_i \eta^i$ is the support $C \subset B$ of the wrapped D3-brane, and $m = 2J$ is twice the angular momentum of the resulting F-theory spinning string.

The microscopic entropy of such a D-brane system is determined by counting cohomology classes on the moduli space of classical supersymmetric configurations. From a mathematical point of view, supersymmetric D-brane configurations correspond to semi-stable coherent sheaves on $X$ with fixed Chern classes given by (2.15). In general the geometry of moduli spaces of semi-stable coherent sheaves is very little understood on Calabi-Yau threefolds. These spaces are expected to have very complicated singularities which make a mathematical formulation of the counting problem very difficult.

The D6-D2-D0 configurations considered in this section can however be mapped to D4-D2-D0 configurations by the duality chain of section two. We will show below that this map is in fact a relative Fourier-Mukai transform along the elliptic fibers. Then the counting problem becomes more tractable, and we can employ the methods of [5, 6, 19] in order to determine the asymptotic growth of the microstates in the limit of large D2-brane charge on the elliptic fiber.

The physical applications of the Fourier-Mukai transform have been focused so far on heterotic bundle constructions and heterotic-F-theory duality starting with the work of [46–50]. It has also been considered in [51, 52] in connection with homological mirror symmetry, which is closer to our context. The Fourier-Mukai transform can be intuitively thought of as T-duality along the elliptic fibers. However naive T-duality is not well defined in the presence of singular elliptic fibers, hence we have to employ a more sophisticated transformation which is defined abstractly as a derived functor. Since the technical details have been thoroughly worked out in the above papers, we will only recall the essential facts omitting most technical details. It is worth noting however that the Fourier-Mukai transform is not an element of the T-duality group of the theory, which is generated by monodromy transformations acting on the derived category [53, 54]. This question was investigated in detail in [51, 52], where it was found that the Fourier-Mukai transform differs from a monodromy transformation by a certain twist. This agrees with the transformation found in section two, which involved nonperturbative duality transformations.

The Fourier-Mukai transform of the D6-D2-D0 configuration described by a bundle $\mathcal{E}$ is a D4-D2-D0 system described by a derived object $\mathcal{F}[1]$, where $\mathcal{F}$ is a torsion sheaf $\mathcal{F}$ on $X$ supported on a divisor $\Sigma \subset X$. The effect of the shift by 1 is to change the sign of all D-brane charges of the configuration represented by the sheaf $\mathcal{F}$. Moreover, according to [55], the Fourier-Mukai transform preserves semi-stability with respect to a suitable polarization of $X$. This means it maps supersymmetric D-brane configurations to supersymmetric D-brane configurations, therefore we can reliably use it in order to count BPS states.

The Chern character of $\mathcal{F}$ is [51]

$$
\begin{aligned}
\mathrm{ch}_1(\mathcal{F}) &= r\sigma + \pi^* C \\
\mathrm{ch}_2(\mathcal{F}) &= -\frac{r}{2}\sigma \wedge \pi^* c_1(B) + \left( m + \frac{1}{2}\int_X \sigma \wedge \pi^* c_1(B) \wedge \pi^* C \right)\beta \\
\mathrm{ch}_3(\mathcal{F}) &= -n\omega + \frac{r}{6}\sigma \wedge \pi^* c_1(B)^2.
\end{aligned}
\tag{2.18}
$$

where

$$
C = q_i \eta^i.
\tag{2.19}
$$

Note that $C$ can be interpreted by Poincaré duality as a curve class on $B$. We will assume that $C$ is a very ample divisor class on $B$ of sufficiently high degree so that the generic surface $\Sigma$ in the class $r\sigma + \pi^* C$ is smooth and irreducible.

One can easily check that the above of the Fourier-Mukai transform on topological charges is in agreement with the duality map outlined in section (2.1) up to corrections involving $c_1(B)$. This is positive evidence for the identification of these two transformations. The curvature corrections are not under control in the chain of dualities described in section two, hence we will not be able to perform a more detailed check. We will obtain more compelling evidence by matching the black hole entropy formulas in the next section.

The leading contribution to the entropy of a D4-D2-D0 configuration in the limit of large D0 charge has been evaluated in [5, 6, 19]. As a first step, we need to identify the BPS charges $(\widetilde{P}^A, \widetilde{Q}_A)$ of this configuration by computing the leading terms of the central charge

$$
Z(\mathcal{F}[1]) = -Z(\mathcal{F})
\tag{2.20}
$$

near the large radius limit point. More precisely, we have to express

$$
\begin{aligned}
Z(\mathcal{F}) &= \int_X e^J \mathrm{ch}(\mathcal{F})\sqrt{Td(X)} \\
&= \int_X \left[ \frac{1}{2}J^2 \mathrm{ch}_1(\mathcal{F}) + J\mathrm{ch}_2(\mathcal{F}) + \frac{1}{2}\mathrm{ch}_1(\mathcal{F})\mathrm{Td}_2(X) + \mathrm{ch}_3(\mathcal{F}) \right]
\end{aligned}
\tag{2.21}
$$

as a linear combination of the functions $Z(\alpha_A)$, $Z(\beta^A)$ which appear in the expansion (2.9) of the periods at the large radius limit point.

For this computation we will need the triple intersection numbers

$$
\begin{aligned}
D_{hhh} &= \frac{1}{6}\int_X \alpha_h^3 = \frac{1}{6}\int_X (\sigma + \pi^* c_1(B))^3 = \frac{1}{6}\int_B c_1(B)^2 \\
D_{hhi} &= \frac{1}{6}\int_X \alpha_h^2 \wedge \alpha_i = \frac{1}{6}\int_X \sigma \wedge \sigma \wedge \alpha_i = \frac{1}{6}\int_B c_1(B) \wedge \gamma_i \\
D_{hij} &= \frac{1}{6}\int_X \alpha_h \wedge \alpha_i \wedge \alpha_j = \frac{1}{6}\int_X \sigma \wedge \alpha_i \wedge \alpha_j = \frac{1}{6}\int_B \gamma_i \wedge \gamma_j \\
D_{ijk} &= \frac{1}{6}\int_X \alpha_i \wedge \alpha_j \wedge \alpha_k = 0.
\end{aligned}
\tag{2.22}
$$

Let us introduce the following notation

$$
d = \int_B c_1(B)^2 \qquad c_i = \int_B c_1(B) \wedge \gamma_i \qquad d_{ij} = \int_B \gamma_i \wedge \gamma_j.
\tag{2.23}
$$

Note that we have

$$\int_B \eta^i \wedge \eta^j = d^{ij} \qquad \int_B c_1(B) \wedge \eta^i = d^{ij}c_j \qquad \text{where} \qquad d_{ik}d^{kl} = \delta_i^l. \tag{2.24}$$

We will also make frequent use of the following expressions

$$Q = \int_B C \wedge C = q_i d^{ij} q_j \qquad c = \int_B C \wedge c_1(B) = q_i d^{ij} c_j. \tag{2.25}$$

The Chern character $\text{ch}(\mathcal{F})$ written in terms of the bases $\{\alpha_A\}$ and $\{\beta^A\}$ reads

$$\text{ch}(\mathcal{F}) = r\alpha_h + (q_i - rc_i)d^{ij}\alpha_j + \left(m + \frac{c}{2}\right)\beta^h - \frac{r}{2}c_i\beta^i - \left(n - \frac{rd}{6}\right)\omega. \tag{2.26}$$

Now we substitute equation (2.26) in (2.21) obtaining

$$Z(\mathcal{F}) = \int_X \left[ \frac{1}{2}J^2(r\alpha_h + (q_i - rc_i)d^{ij}\alpha_j)] + \left(m + \frac{c}{2}\right)J\beta^h - \frac{r}{2}c_i J\beta^i \right.$$
$$\left. - (n - \frac{rd}{6})\omega + \frac{1}{2}(r\alpha_h + (q_i - rc_i)d^{ij}\alpha_j)\text{Td}_2(X) \right] \tag{2.27}$$

In terms of the functions $Z(\alpha_i), Z(\alpha_h), Z(\beta^i), Z(\beta^h)$, (2.27) reads

$$Z(\mathcal{F}) = rZ(\alpha_h) + (q_i - rc_i)d^{ij}Z(\alpha_j) + \left(m + \frac{c}{2}\right)Z(\beta^h) - \frac{r}{2}c_i Z(\beta^i) - \left(n - \frac{rd}{6}\right). \tag{2.28}$$

From this expression we can easily read off the charges of the D4-D2-D0 system described by $\mathcal{F}$. Taking into account the sign in equation (2.20), we can now read off the charge vector of the corresponding BPS state

$$\left(\widetilde{P}^0, \widetilde{P}^A, \widetilde{Q}_A, \widetilde{Q}_0\right) = \left(0, (q_i - rc_i)d^{ij}, r, -rc_i/2, m + c/2, n - rd/6\right). \tag{2.29}$$

Note that the coefficients in the above expansion are not in general integral. In principle one can choose a different basis of periods which makes the integrality of charges manifest. Since this is not a very important point for the following computations, we postpone this discussion for appendix A.

According to [5], the asymptotic microstate degeneracy of the D4-D2-D0 system in the limit of large D0 charge is determined by the degeneracy of states in a $(0, 4)$ CFT obtained by lifting the system to M theory. The left moving central charge of the CFT is given by

$$c_L = D + \frac{1}{6}\int_X (r\sigma + \pi^*C) \wedge c_2(X) \tag{2.30}$$

where

$$D = \frac{1}{6}\int_X (r\sigma + q_i\pi^*\eta^i)^3$$
$$= \frac{1}{6}\left(dr^3 - 3r^2 q_i c_i d^{ij} + 3r q_i q_j d^{ij}\right) \tag{2.31}$$
$$= \frac{1}{6}(dr^3 - 3r^2 c + 3rQ).$$

The microstate degeneracy is determined by the asymptotic growth of states of momentum

$$\widehat{m} = n + \frac{1}{12}(D^{\alpha\alpha}\widetilde{Q}_\beta\widetilde{Q}_\beta + 2D^{\alpha i}\widetilde{Q}_\beta\widetilde{Q}_i + D^{ij}\widetilde{Q}_i\widetilde{Q}_j) \tag{2.32}$$

where

$$\begin{bmatrix} D^{\alpha\alpha} & D^{\alpha i} \\ D^{i\alpha} & D^{ij} \end{bmatrix} = \begin{bmatrix} D_{\alpha\alpha} & D_{\alpha i} \\ D_{i\alpha} & D_{ij} \end{bmatrix}^{-1}.$$

and

$$D_{\alpha\alpha} = D_{\alpha\alpha\alpha}\widetilde{P}^\alpha + D_{\alpha\alpha i}\widetilde{P}^i = \frac{1}{6}(rd + (q_i - rc_i)d^{ij}c_j) = \frac{1}{6}q_i c_j d^{ij}$$

$$D_{\alpha i} = D_{\alpha i\alpha}\widetilde{P}^\alpha + D_{\alpha ij}\widetilde{P}^j = \frac{1}{6}(rc_i + d_{ij}(q_k - rc_k)d^{kj}) = \frac{1}{6}q_i \tag{2.33}$$

$$D_{ij} = D_{ij\alpha}\widetilde{P}^\alpha = \frac{1}{6}rd_{ij}$$

Applying Cardy's formula, we find the leading term in the entropy formula to be

$$S_{\text{micro}} = 2\pi\sqrt{D\widehat{m}}. \tag{2.34}$$

Note that this formula captures the microstate degeneracy due to a gas of $\widehat{m}$ D0-brane bound to a fixed D4-brane wrapping a divisor $\Sigma$ in the class $(r\sigma + \pi^*C)$ [19]. In particular, this is not an exact formula for the entropy of the D4-D2-D0 configuration, and it does not capture the asymptotic behavior at large $r$. In order to capture the later behavior one has to integrate on the moduli space of the D4-brane, which is a very difficult computation. We leave this issue for later work.

# 3. Six dimensional black strings and macroscopic entropy

The purpose of this section is to find the macroscopic description of the brane configurations discussed in sections two and three in terms of low energy supergravity. We will first show that the four or five dimensional attractor mechanism does not yield reliable solutions in the limit required by Cardy's formula. We will also show that a reliable low energy description of the system must be formulated in terms of black string solutions of six dimensional $N = 1$ supergravity. The black string entropy will be shown to agree with the leading behavior of the microscopic result (2.34) in the limit of large charges.

## 3.1 D6-D2-D0 attractors

Let us first try to solve the attractor equations [56–58] for black holes carrying D6-D2-D0 charges in a neighborhood of the large radius limit point in the Kähler moduli space. According to [29, 42], this is equivalent to solving five dimensional attractor equations for the dual M2-brane configurations.

Following [59, 58], we write the attractor equations in the form

$$iP^\Lambda = Y^\Lambda - \overline{Y}^\Lambda$$
$$iQ_\Lambda = \mathcal{F}_\Lambda(Y) - \overline{\mathcal{F}}_\Lambda(Y) \tag{3.1}$$

where the new variables $Y^\Lambda$ are defined by

$$Y^\Lambda = \overline{\mathcal{Z}} X^\Lambda.$$

Here $\mathcal{Z}$ denotes the central charge of a BPS states with charges $(p^\Lambda, q_\Lambda)$ (2.5). The macroscopic entropy is given by

$$S_{\text{macro}} = i\pi \left( \overline{Y}^\Lambda \mathcal{F}_\Lambda(Y) - Y^\Lambda \overline{\mathcal{F}}_\Lambda(Y) \right). \tag{3.2}$$

In our case the charge vector is given by (2.17), hence the equations (3.1) reduce to

$$
\begin{aligned}
iP^0 &= Y^0 - \overline{Y}^0 & iQ_0 &= \mathcal{F}_0(Y) - \overline{\mathcal{F}}_0(Y) \\
0 &= Y^A - \overline{Y}^A & iQ_A &= \mathcal{F}_A(Y) - \overline{\mathcal{F}}_A(Y)
\end{aligned}
\tag{3.3}
$$

The solution of these equations is of the form [58, 60]

$$
\begin{aligned}
S_{\text{macro}} &= \frac{\pi}{3P^0} \sqrt{\frac{4}{3}(\Delta_A y^A)^2 - 9((P^0)^2 Q_0)^2} \\
t^A &= \frac{3}{2} \frac{y^A}{\Delta_A y^A} (P^0 Q_0) - i \frac{3}{2} \frac{y^A}{\Delta_A y^A} \frac{S_{\text{macro}}}{\pi}
\end{aligned}
\tag{3.4}
$$

where $y^A$ are solutions to the quadratic equations

$$D_{ABC} y^A y^B = \Delta_C, \qquad \Delta_C = -P^0 Q_C. \tag{3.5}$$

An existence condition for the attractor point is that the solutions $y^A$ of (3.5) be real. Moreover the attractor solution is self-consistent only if the imaginary parts $\text{Im}(t^A)$ of the Kähler parameters in (3.4) are large and negative.

Next let us specialize equations (3.3) to D6-D2-D0 configurations on elliptic Calabi-Yau threefolds. In this case, the charge vector is given in equation (2.17). We find that the entropy formula is given by

$$S_{\text{macro}} = \frac{\pi}{3r} \sqrt{\frac{4}{3}(\Delta_h y^h + \Delta_i y^i)^2 - 9r^4 m^2} \tag{3.6}$$

where

$$\Delta_h = rn \qquad \Delta_i = rq_i \tag{3.7}$$

and $y^i, y^h$ are solutions of the system of quadratic equations

$$
\begin{aligned}
D_{hhh}(y^h)^2 + 2D_{hhi} y^h x^i + D_{ijh} y^i y^j &= rn \\
D_{hhi}(y^h)^2 + 2D_{ijh} y^h y^j &= rq_i.
\end{aligned}
\tag{3.8}
$$

Using formulas (2.22), (2.23), equations (3.8) become

$$
\begin{aligned}
\frac{1}{6} d(y^h)^2 + \frac{1}{3} c_i y^h y^i + \frac{1}{6} d_{ij} y^i y^j &= rn \\
\frac{1}{6} c_i (y^h)^2 + \frac{1}{3} d_{ij} y^h y^j &= rq_i
\end{aligned}
\tag{3.9}
$$

Using the linear equations in the $y^i$, we find

$$y^i = d^{ij} \left( \frac{3rq_j}{y^h} - \frac{c_j y^h}{2} \right) \tag{3.10}$$

Substituting equations (3.10) in the first equation in (3.9) we obtain the quartic equation

$$\frac{d}{24}(y^h)^2 + \frac{r}{2}q_i d^{ij} c_j + \frac{3r^2}{2}q_i d^{ij} q_j (y^h)^{-2} = rn. \tag{3.11}$$

Using the notations (2.25), we can rewrite equation (3.11) in the final form

$$\frac{d}{24}(y^h)^4 - r\left(n - \frac{c}{2}\right)(y^h)^2 + \frac{3r^2 Q}{2} = 0. \tag{3.12}$$

Solving for $(y^h)^2$, we find

$$(y^h_\pm)^2 = \frac{12r}{d} \left[ \left(n - \frac{c}{2}\right) \pm \sqrt{\left(n - \frac{c}{2}\right)^2 - \frac{dQ}{4}} \right] \tag{3.13}$$

Using equations (3.10), the macroscopic entropy formula (3.6) can be expressed as a function of $y^h$ as follows

$$S_{\text{macro}} = \frac{\pi}{3} \sqrt{ \frac{4}{3} \left[ \left(n - \frac{c}{2}\right) y^h_\pm + \frac{3rQ}{y^h_\pm} \right]^2 - 9r^2 m^2 } \tag{3.14}$$

The values of the Kähler moduli at the attractor point are given by

$$\begin{aligned} t^h &= \frac{3rm y^h}{2\Delta} - i \frac{3y^h S_{\text{macro}}}{2\pi\Delta} \\ t^i &= \frac{3rm y^i}{2\Delta} - i \frac{3y^i S_{\text{macro}}}{2\pi\Delta} \end{aligned} \tag{3.15}$$

where

$$\Delta = \left(n - \frac{c}{2}\right) y^h_\pm + \frac{3rQ}{y^h_\pm} \tag{3.16}$$

Now let us review the regime of validity of the microscopic formula (3.14). We must satisfy the following conditions

i) The curve class $C = q_i \eta^i$ should be sufficiently ample on $B$ so that $\Sigma = r\sigma + \pi^* C$ is a very ample divisor on $X$. More precisely, a generic surface $\Sigma$ is smooth and irreducible if $C$ is an effective curve class on $B$ and also $C - c_1(B)$ is a smooth irreducible curve on $B$ [61]. If we choose the basis elements $\eta^i$, $i = 1, \ldots, h^{1,1}(B)$, to be Poincaré dual to generators of the Mori cone, the first condition implies that the integers $q_i$ must be positive. The second condition implies that $q_i > rc_i$ for all $i = 1, \ldots, h^{1,1}(B)$. Note that Cardy's formula for the entropy becomes more and more reliable as we increase $q_i$ keeping $r$ fixed because the divisor $\Sigma$ becomes more and more ample. Then one can neglect the effect of singular divisors in the linear system $|\Sigma|$ on the target space geometry of the $(0,4)$. As shown in [62], the $(0,4)$ sigma model for the

M5-brane is quite involved, and the effects of singular divisors are not under analytic control. We expect these effects to become important for values of $q_i$ comparable to $r$. In particular, if $q_i < rc_i$, the divisor $\Sigma$ is not smooth, and the $(0,4)$ description employed in [5] breaks down.

ii) Assuming condition $(i)$ to be satisfied, validity of Cardy's formula also requires the momentum of the CFT states to be much larger than the central charge. This condition is satisfied if the D0-brane charge $n$ is much larger than $D = \Sigma^3$. From the point of view of the D4-D2-D0 configuration discussed in section three, this means that the formula (2.34) captures the asymptotic behavior of the microstate degeneracy in the limit of large $n$ keeping $r, q_i$ fixed.

The two solutions found in (3.13) have the following leading order behavior in the limit of large $n$, with $r, q_i$ fixed

$$(y_+^h)^2 \sim \frac{24rn}{d} \qquad (y_-^h)^2 \sim \frac{3Q}{2n}. \tag{3.17}$$

One can rule out the first solution observing that for any choice of the sign for $y_h$ at least one of the attractor Kähler parameters is large and positive.[2] This is incompatible with a physical interpretation of the solution, since it would require a large negative volume of the Calabi-Yau threefold.

For the second solution, the leading term of the macroscopic entropy formula is

$$S_{\text{macro}} \sim \pi \sqrt{2rnQ - r^2m^2} \tag{3.18}$$

and the leading behavior of the Kähler moduli at the attractor point is

$$\begin{aligned} \text{Im}(t^h) &\sim -\sqrt{\frac{rQ}{2n}} \\ \text{Im}(t^i) &\sim -d^{ij}\left[ q_j\sqrt{\frac{2n}{rQ}} - \frac{c_j}{2}\sqrt{\frac{rQ}{2n}} \right]. \end{aligned} \tag{3.19}$$

Clearly, for large $n$, $\text{Im}(t^i)$ are very large and negative while $\text{Im}(t^h)$ is negative but very small. Such points do not lie in the neighborhood of the large radius limit of the Kähler moduli space, hence the attractor solution is not self-consistent. One may wonder if a self-consistent attractor solution may exist in other regions of the moduli space. The quantum special geometry of the Kähler moduli space has been solved for the elliptic fibration over $\mathbb{P}^2$ in [63]. Their results show that there is no region in the moduli space where the quantum area of the elliptic fiber is much smaller than the quantum area of a horizontal curve. This does not logically rule out the existence of attractor points in quantum phases of the moduli space, but it suggests that this would not be a natural solution to our problem.

In the following we would like to propose another resolution of this problem suggested by the duality chain of section two. Note that according to [64], IIA compactifications

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[2]We thank F. Denef and G. Moore for discussions on this point.

on elliptic fibrations are equivalent to six dimensional F-theory compactifications on the base in the limit of small elliptic fibers. In this limit, a D6-D2-D0 configuration is mapped to a D3-brane wrapping a holomorphic curve in the base, as discussed in section two. The resulting noncritical string also wraps a transverse circle $S^1$ whose radius is inversely proportional with the size if the elliptic fiber. Therefore the scaling behavior of the Kähler parameters at the attractor point suggests that the correct low energy description of our system should be formulated in terms of black string solutions of $N = 1$ six dimensional supergravity.

### 3.2 Black strings in $N = 1$ supergravity

Let us start with a brief review of F-theory compactifications to six dimensions from the low energy point of view. Since we are interested only in compactifications on smooth Weierstrass models $X \to B$, we have to take $B$ to be a smooth Fano surface, i.e. a del Pezzo surface. Therefore $B$ can be either a $k$-point blow-up of $\mathbb{P}^2$, $0 \le k \le 8$ or $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. For future reference we will choose a basis $\{\gamma_i\}$ of $H^{1,1}(B)$ of the form

$$
\begin{array}{llllll}
\gamma_1 = e_1 & \gamma_2 = e_2 & \ldots & \gamma_k = e_k & \gamma_{k+1} = h & \text{for } B = dP_k \\
\gamma_1 = a & \gamma_2 = b & & & & \text{for } B = \mathbb{F}_0
\end{array}
\tag{3.20}
$$

where $h$ denotes the hyperplane class of $\mathbb{P}^2$ and $e_1, \ldots, e_k$ denote the exceptional curve classes. In the case $B = \mathbb{F}_0$, $a, b$ denote the classes of the two rulings. The dual basis $\{\eta^i\}$ is

$$
\begin{array}{llllll}
\eta^1 = -e_1 & \eta^2 = -e_2 & \ldots & \eta^k = -e_k & \eta^{k+1} = h & \text{for } B = dP_k \\
\eta^1 = b & \eta^2 = a & & & & \text{for } B = \mathbb{F}_0
\end{array}
\tag{3.21}
$$

The intersection matrix $(d_{ij})$ reads

$$
\begin{aligned}
(d_{ij}) &= \text{diag}(-1, -1, \ldots, -1, 1) & \text{for } B = dP_k \\
(d_{ij}) &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{for } B = \mathbb{F}_0
\end{aligned}
\tag{3.22}
$$

Let us first consider the case $B = dP_k$, $0 \le k \le 8$, leaving $B = \mathbb{F}_0$ for a separate discussion. The low energy supergravity theory contains a $N = 1$ graviton multiplet and $k = (h^{1,1}(B) - 1)$ $N = 1$ tensor multiplets. The bosonic spectrum consists of the metric tensor, $(k + 1)$ elementary tensor multiplets and $k$ real scalar fields. The tree level formulation of the theory has been described in detail in [65]. The scalar components of the tensor multiplets take values in the coset manifold $O(k, 1)/O(k)$. They are parameterized by an $O(k, 1)$ valued field

$$
V(x) = \begin{bmatrix} x_b^a & x_{k+1}^a \\ v_a & v_{k+1} \end{bmatrix}
$$

where $a, b = 1, \ldots, k$ subject to local $O(k)$ gauge transformations

$$
V(x) \to g(x) V(x), \qquad g(x) \in O(k)
$$

and global $SO(1,k)$ symmetry transformations

$$V(x) \to V(x)U^{-1}, \qquad U \in O(k,1).$$

The $(k+1)$ elementary antisymmetric tensor fields $B^1, \ldots, B^{k+1}$ are obtained by Kaluza-Klein reduction of the type IIB four-form potential $C^{(4)}$ on the basis $\{\gamma_i\}$ of harmonic $(1,1)$ forms

$$C^{(4)} = \sum_{i=1}^{k+1} B^i \wedge \gamma_i. \tag{3.23}$$

The tensor fields $B = B^i$ transform in the fundamental representation of the global symmetry group $O(k,1)$,

$$B^i \to U_j^i B^j, \qquad U \in O(k,1),$$

and are subject to certain self-duality constraints formulated in terms of the $O(k,1)$-invariant tensor fields

$$K^a = x_b^a dB^b + x_{k+1}^a dB^{k+1} \qquad H = v_a dB^a + v_{k+1} dB^{k+1}.$$

$H$ is required to be self-dual and the $K^a$, $a = 1, \ldots, k$ are required to be anti-self-dual. The expectation values of the scalar components $v_a$, $a = 1, \ldots, k$ and $v_{k+1}$ are related to the Kähler moduli of the F-theory base. This follows from the fact that the space $H_+^{1,1}(B)$ of self-dual $(1,1)$ harmonic forms is spanned by the Kähler class $J_B$. The space of anti-self-dual $(1,1)$ harmonic forms is the orthogonal complement of $J_B$ in $H^{1,1}(B)$. Let us write the Kähler class of $B$ as

$$J_B = t_i \eta^i$$

where $t^i$ are real valued Kähler moduli. Then we have

$$H = \frac{1}{2\text{vol}(B)} \, t_i dB^i.$$

Note that the volume of the base is parameterized by the expectation value of a scalar component of a six dimensional hypermultiplet, therefore the $t_i$ will be subject to a constraint of the form

$$d^{ij} t_i t_j = 2\text{vol}(B) = \text{constant} \tag{3.24}$$

which is reminiscent of the more familiar cubic constraint in five dimensional supergravity. By rescaling the fields we may take this constant to be 1. Then we can identify $t^i = v^i$, $i = 1, \ldots, k+1$, and the constraint is part of the the orthogonality condition

$$\eta V^T \eta = V^{-1} \tag{3.25}$$

where $\eta$ is the Minkowski metric tensor of signature $(k,1)$.

For future reference, let us consider the case $k = 1$ in more detail. In this case, the field $V$ can be chosen of the form [65]

$$V = \begin{bmatrix} \cosh(\phi) & \sinh(\phi) \\ \sinh(\phi) & \cosh(\phi) \end{bmatrix} \tag{3.26}$$

and the self-dual and anti-self-dual field strengths are given by

$$H = \cosh(\phi)B^2 + \sinh(\phi)B^1$$
$$K = \cosh(\phi)B^1 + \sinh(\phi)B^2.$$

This allows us to identify the Kähler parameters of the base $B = \mathbb{F}_1$ as

$$t_1 = \sinh(\phi) \qquad t_2 = \cosh(\phi). \tag{3.27}$$

Note that in the case $k = 1$ we can give a conventional lagrangian formulation of the theory in terms of either $B^1 + B^2$ or $B^1 - B^2$ regarded as an unconstrained tensor fields.

The above considerations are valid for $B = dP_k$, $0 \leq k \leq 8$. The case $B = \mathbb{F}_0$ also results in a low energy effective action with one tensor multiplet, which has the same tree level formulation as the case $B = \mathbb{F}_1$. The main difference between $\mathbb{F}_0$ and $\mathbb{F}_1$ resides in the relation between the Kaluza-Klein zero modes of $C^{(4)}$ and the elementary tensor fields $B^1, B^2$. In this case we have

$$C^{(4)} = \left[ B^1 \wedge \frac{b-a}{\sqrt{2}} + B^2 \wedge \frac{b+a}{\sqrt{2}} \right]. \tag{3.28}$$

The theory can be alternatively formulated in terms of the Kaluza-Klein modes $C^1, C^2$ defined with respect to the natural basis $\{a, b\}$ of $(1,1)$ forms on $\mathbb{F}_0$ given by the two rulings,

$$C^{(4)} = C^1 a + C^2 b. \tag{3.29}$$

Note that

$$C^1 = \frac{B^2 - B^1}{\sqrt{2}} \qquad C^2 = \frac{B^2 + B^1}{\sqrt{2}}. \tag{3.30}$$

Either $C^1$ or $C^2$ can be regarded as unconstrained tensor fields, leading to a conventional lagrangian formulation of the theory. Moreover, the Kähler class has the form

$$J_B = t_1 b + t_2 a$$

where

$$t_1 = \frac{1}{\sqrt{2}} e^\phi \qquad t_2 = \frac{1}{\sqrt{2}} e^{-\phi}. \tag{3.31}$$

The black strings we are interested in are obtained by wrapping D3-branes on curves of the form $C = q_i \eta^i$ in $B$, which are charged with respect to the elementary tensor fields $B^1, B^2$. The charge lattice is

$$\Gamma_B \simeq H^2(B, \mathbb{Z})$$

equipped with the symmetric bilinear form defined in (3.22). Charge quantization breaks the global $O(k,1)$ symmetry group to an integral subgroup $\text{Aut}(\Gamma_B) \subset O(k,1)$. In addition, these strings carry $n$ units of KK momentum on circle $S^1$ transverse to $B$ and have angular momentum $J$. The extra charges $(n, J)$ are invariant under U-duality transformations.

In order to compute the macroscopic energy we have to find supersymmetric black string solutions of $N = 1$ six dimensional supergravity with charges $(\underline{q}, n, J)$. These solutions have been completely classified for for the minimal theory (i.e. $k = 0$) in [66] and

for gauged supergravity with one tensor multiplet (i.e. $k = 1$) in [67]. One can also obtain the results in the ungauged case either by adapting the results of [67] to the ungauged case, or, as we will show below, by dualizing solutions of $U(1)^3$ ungauged supergravity in 5 dimensions. Analogous results for higher numbers of tensor multiplets do not seem to be available in the literature, but an exhaustive classification is not really needed for our purposes.

A very useful observation is that U-duality transformations, which correspond to automorphisms of the charge lattice, map supergravity solutions to supergravity solutions preserving the entropy. Therefore for any $k \geq 2$ we can reduce the problem to $k = 1$ as long as the charge vector

$$\underline{q} = q_i \eta^i \tag{3.32}$$

can be mapped by a U-duality transformation to a charge vector contained in a $(1, 1)$ sublattice. For the type of lattices under consideration, this is not always the case [68], but we will restrict ourselves only to such charge vectors from now on. A similar argument was previously used in a similar context in [7]. Without loss of generality we can take the $(1, 1)$ sublattice to be spanned by $(h, e_1)$.

The case $B = \mathbb{F}_0$ can be easily solved observing that the resulting $N = 1$ theory expressed for example in terms of the unconstrained field $C^2$ is identical to a subsector of the extended $N = 4$ supergravity obtained by reduction of the IIB theory on $T^4$. The bosonic components of the subsector in question are the metric tensor, the six dimensional reduction of the RR two-form potential $C$ and the dilaton field $\phi$. We will refer to this truncation as the D1-D5 subsector since these are precisely the fields which couple with six dimensional D1-D5 strings. The identification of these two models is justified by the isomorphism

$$
\begin{aligned}
H^{1,1}(\mathbb{F}_0) &\xrightarrow{\simeq} H^0(T^4) \oplus H^4(T^4) \\
(a, b) &\longrightarrow (1, w)
\end{aligned}
\tag{3.33}
$$

where $w$ is a generator of $H^4(T^4)$ normalized so that $\int_{T^4} w = 1$. This isomorphism is compatible with the bilinear intersection forms. Then one can check that the two low energy effective actions are identical if we identify $C^2$ with the RR two-form $C$ and the field $e^\phi$ introduced in (3.26) with the dilaton field. Note that this is only a formal identification of the tree level supergravity actions. It does not imply that the two physical theories are equivalent, which is clearly not the case, but it is a useful technical tool in writing down supergravity solutions. In particular note that although the low energy fields are formally identified, they have very different interpretations in the two theories. For example $e^\phi$ is the dilaton field in the IIB theory on $T^4$, while it is related to the Kähler parameters of the base in F-theory on $\mathbb{F}_0$. In the following we will think of the D1-D5 subsector of IIB supergravity on $T^4$ just as an auxiliary model with no direct physical relevance.

The identification observed in the last paragraph is useful because now one can simply reinterpret the six dimensional solution for a D1-D5 string on $T^4$ as a black string solution in the F-theory compactification. In particular, the charges of the two solutions are related

by

$$Q_1 = q_2 \qquad Q_5 = q_1. \tag{3.34}$$

where $q_1, q_2$ are the black string charges with respect to the tensor fields $C^1, C^2$ defined in (3.29)

Since the $\mathbb{F}_1$ theory is related at tree level to the $\mathbb{F}_0$ by a field redefinition given in equations (3.28), that we can obtain similarly black string solutions for F-theory on $\mathbb{F}_1$. In this case the charges should be related as follows

$$Q_1 = \frac{q_2 + q_1}{\sqrt{2}} \qquad Q_5 = \frac{q_2 - q_1}{\sqrt{2}}. \tag{3.35}$$

Note that $Q_1, Q_5$ need not be integral since we are only using the six dimensional tree level supergravity solution of the D1-D5 system as convenient technical tool. At this level, we can simply regard $Q_1, Q_5$ as continuous parameters of the solution.

The case $B = \mathbb{P}^2$ is somewhat special since it leads to minimal $N = 1$ supergravity without tensor multiplets. In fact a black string solution in the $\mathbb{P}^2$ theory can be regarded as a similar solution in the $\mathbb{F}^1$ theory with $q_1 = 0$. Therefore it will be obtained from the D1-D5 solution setting

$$Q_1 = Q_5 = \frac{q}{\sqrt{2}} \tag{3.36}$$

where $\underline{q} = qh$ is the charge vector of the F-theory black string.

As explained in section two, in our case the black strings wrap a circle of radius $R$, and are also transverse to a Taub-NUT space. In addition to the charges $\underline{q}$ they also carry $n$ units of KK momentum on this circle and have an angular momentum $J$. [3] The formal identification of the corresponding supergravity solutions to the solution of a D1-D5 system can be trivially extended to this case. We will need therefore to find solutions for a D1-D5 system in an identical six dimensional background geometry with the same KK momentum $n$ on the circle and the same angular momentum $J$.

This solution can in fact be obtained by dualizing a five-dimensional supergravity solution corresponding to M-theory on $T^6 \times TN_r \times R$ with $Q_1, Q_5$ and $n$ M2 branes wrapping three orthogonal two-cycles in $T^6$ (see for example [30, 69]). Such five-dimensional supergravity solutions have been classified and studied in much detail in [70−72][4] and the explicit T-duality transformation can be found for example in [30].

Let $u$ denote an angular coordinate on $S^1$ with periodicity $u \sim u + 2\pi R$ and let $(\psi, x^1, x^2, x^3)$ denote coordinates on the Taub-NUT space of charge $r$. The angular coordinate $\psi$ has periodicity $\psi \sim 4\pi r$, and $(x^1, x^2, x^3)$ are cartesian coordinates on the $\mathbb{R}^3$ base of Taub-NUT. The metric is

$$ds^2_{TN_r} = V d\vec{x}^2 + V^{-1}(d\psi + \vec{A}d\vec{x})^2 \tag{3.37}$$

---

[3] When the transverse space is Taub-NUT, the quantity $J$ is not strictly speaking an angular momentum. However, if one replaces the transverse Taub-NUT by $\mathbb{R}^4$ (or if one zooms in near the center of a Taub-NUT space of charge one to recover a solution in $\mathbb{R}^4$) this string becomes the six-dimensional lift of a five-dimensional BMPV black hole with angular momenta $J_1 = J_2 = J$. When the transverse space is Taub-NUT, the four-dimensional interpretation of the quantity $J_1 + J_2 = 2J = m$ is that of D0 charge, or KK momentum charge along the Taub-NUT circle.

[4] See [32, 31, 33, 73−75, 3, 4, 76−83] for related studies.

where

$$V = h + \frac{r}{|\vec{x}|}$$

and

$$\nabla \times \vec{A} = \nabla V.$$

The constant $h$ is a modulus that is inversely proportional to the radius of the circle fiber at infinity. When $h = 0$ the metric (3.37) becomes that of $\mathbb{R}^4$, and the radius in $\mathbb{R}^4$ is related to $\vec{x}$ via $r_{\mathbb{R}^4}^2 = 4|\vec{x}|$.

The six dimensional metric and background fields depend on four harmonic functions [30, 72]

$$Z_1 = 1 + hc_1 + \frac{Q_1}{4|\vec{x}|} \qquad Z_5 = 1 + hc_5 + \frac{Q_5}{4|\vec{x}|}$$
$$Z_p = 1 + hc_p + \frac{n}{4|\vec{x}|} \qquad Z_J = hc_J + \frac{J}{4|\vec{x}|} \tag{3.38}$$

on the Taub-NUT space, where $Q_1, Q_5$ are D1 and D5 charges respectively, $n$ is the KK momentum along the circle, $J$ is the "5D angular momentum" that corresponds to KK momentum along the Taub-NUT direction, and the parameters $c_1, c_5, c_p$ and $c_J$ are moduli of the solution. We work in a convention in which $G_6 = \frac{\pi}{4} 2\pi R$, and in which the charges that appear in the supergravity solution are the same as the quantized D-brane charges. It is easy to see than when $h$ is set to zero, these harmonic functions become the harmonic functions that give the BMPV black hole with angular momenta $J_1 = J_2 = J$.

In order to write down the metric, let us construct a one-form

$$\omega = \frac{Z_J}{2}(d\psi + \vec{A}d\vec{x}) + \vec{\omega}d\vec{x}$$

on the Taub-NUT space, where $\vec{\omega}$ depends only on $\vec{x} \in \mathbb{R}^3$ and is determined by

$$\nabla \times \vec{\omega} = \frac{1}{2}(V\nabla Z_J - Z_J\nabla V). \tag{3.39}$$

The six dimensional metric is of the form

$$ds_6^2 = H^{-1}Z_p du^2 - 2H^{-1}du(dt + \omega) + Hds_{TN}^2 \tag{3.40}$$

where

$$H \equiv (Z_1 Z_5)^{1/2}.$$

Note that $\vec{\omega}$ is determined by condition (3.39) only up to a gradient on $\mathbb{R}^3$, which can be absorbed by a redefinition of the time coordinate $t$. Moreover, the field strength $d\omega$ of $\omega$ is anti-self-dual on the Taub-NUT space by construction. The dilaton $e^\phi$ is

$$e^\phi = \left(\frac{Z_1}{Z_5}\right)^{1/2} \tag{3.41}$$

Solutions of the form (3.40), (3.41) can either be obtained by U-duality from a five-dimensional M-theory on $T^6$ (or $U(1)^3$) supergravity solution of the BMPV black hole in

Taub-NUT, and also as $u$-independent non-twisting solutions in the formalism of [66, 67]. They describe a six dimensional black string solutions with a horizon of the form $S^1 \times S^3$. Its macroscopic entropy is

$$S_{\text{IIB}} = 2\pi\sqrt{rnQ_1Q_5 - r^2J^2} \tag{3.42}$$

independent of the values of the moduli $R, h, c_1, c_5, c_p, c_J$. When $h = 0$ and $r = 1$, the Taub-NUT space becomes $\mathbb{R}^4$, and this black string reduces to the six-dimensional lift of the BMPV black hole.

One can also understand the macroscopic entropy (3.42) from a four-dimensional perspective, although as we explained in Subsection 4.1, the values of the moduli at the horizon in the solution of interest make it intrinsically six-dimensional. If one U-dualizes this solution to one where the three charges correspond to M2 branes wrapping the three $T^2$'s of the $T^6$, and then further compactifies the Taub-NUT space along the fiber, one obtains a four-dimensional black hole that has D2 charges $Q_1, Q_2$ and $n$, KK monopole (D6) charge $r$ and KK momentum (D0) charge $m = 2J$. The entropy of this black hole is again given by (3.42) (see for example [9].)

Taking into account the charge identification (3.34), (3.36), it follows that in the cases $B = \mathbb{P}^2, \mathbb{F}_0, \mathbb{F}_1$, the macroscopic entropy of an F-theory black string is given by

$$S_{\text{macro}} = \pi\sqrt{2rnQ - r^2m^2} \tag{3.43}$$

where

$$Q = d^{ij}q_iq_j.$$

As explained in the paragraph containing equation (3.32), the case $B = dP_k$, $2 \le k \le 8$ can be reduced to the case $B = \mathbb{F}_1$ if the charge vector (3.32) can be mapped by an automorphism of $\Gamma_B$ to a $(1,1)$ sub-lattice. Therefore formula (3.43) will hold in those cases as well.

Finally, note that the Kähler moduli of the F-theory base are fixed by an attractor mechanism. For the $\mathbb{F}_0$ model, using equations (3.31), (3.34), we find

$$t_1 = \frac{1}{\sqrt{2}}\frac{q_2}{q_1} \qquad t_2 = \frac{1}{\sqrt{2}}\frac{q_1}{q_2} \tag{3.44}$$

at the attractor point. As expected, these values are independent of the moduli of the solution. For the $\mathbb{F}_1$ model, equations (3.27) and (3.36) yield

$$t_1 = \frac{1}{2}\left(\frac{q_2}{q_1} - \frac{q_1}{q_2}\right) \qquad t_2 = \frac{1}{2}\left(\frac{q_2}{q_1} + \frac{q_1}{q_2}\right). \tag{3.45}$$

Note that the solution is physically sensible only if $t^1, t^2$ are positive. For the $\mathbb{F}_0$ model, this will hold if $q_1, q_2 > 0$ while for the $\mathbb{F}_1$ model we need $q_1, q_2 > 0$, and $q_2 > q_1$. These are precisely the ampleness conditions for the divisor $C = q_i\eta^i$ on $\mathbb{F}_0$ and $\mathbb{F}_1$ respectively. For more general models, the values of the Kähler parameters can be obtained by U-duality transformations.

It is also worth noting that the attractor mechanism also fixes the radius of the circle parameterized by $u$ to

$$n\sqrt{\frac{2}{rQ}}.$$

If we take $n$ much larger than $Q$, the circle is very large at the attractor point, hence the geometry is six dimensional. This is consistent with the behavior of the Kähler parameters of the four dimensional attractor solutions found in the previous subsection.

### 3.3 Comparison with microscopic entropy

Our next goal is to understand the relation between the microscopic entropy formula (2.34) and the macroscopic formula (3.43). Summarizing conditions ($i$) and ($ii$) below (3.16), recall that the microscopic entropy formula is reliable if

$$n >> q_i >> 0, \qquad q_i >> rc_i, \tag{3.46}$$

assuming that $\eta^i$ are generators of the Mori cone of $B$. In this limit we have

$$Q >> c$$

since

$$Q - c = (C \cdot (C + K_B))_B = 2g(C) - 2$$

is the arithmetic genus of $C$, which is very large and positive for very large $q_i$.

Let us examine the behavior of the microscopic entropy (2.34) in this limit. The leading term in the expression of the triple intersection (2.31) is

$$D \sim \frac{rQ}{2}. \tag{3.47}$$

This yields

$$S_{\text{micro}} \sim \pi\sqrt{2rQ\widehat{m}} \tag{3.48}$$

where $\widehat{m}$ is given by (2.32). The leading term of $\widehat{m}$ at large $m$ is given by

$$\widehat{m} \sim n + \frac{1}{12}D^{\alpha\alpha}m^2. \tag{3.49}$$

In order to compute $D^{\alpha\alpha}$, first note that

$$\det \begin{bmatrix} D_{\alpha\alpha} & D_{\alpha i} \\ D_{i\alpha} & D_{ij} \end{bmatrix} = \det\left(\frac{r}{6}d_{ij}\right)\left(\frac{c}{6} - \frac{Q}{6r}\right) \sim -\frac{Q}{6r}\det\left(\frac{r}{6}d_{ij}\right).$$

Then we have

$$D^{\alpha\alpha} \sim -\frac{6r}{Q}$$

and (3.49) becomes

$$\widehat{m} \sim n - \frac{r}{2Q}m^2.$$

Therefore the leading behavior of the microscopic energy (3.48) is

$$S_{\text{micro}} \sim \pi\sqrt{2rnQ - r^2m^2} \tag{3.50}$$

which is identical to the leading behavior of the macroscopic formula (3.43).

### 3.4 Subleading corrections

We conclude this section with a brief discussion of subleading corrections. So far we have taken into account only the leading terms in the expression of the left moving central charge (2.30) in the limit (3.46). There are two types of subleading corrections. One could take into account subleading terms in the expression of the triple intersection (2.31) and the correction terms of the form

$$\frac{1}{6} \int_X (r\sigma + \pi^* C) \wedge c_2(X)$$

to the central charge. Here we will concentrate only on the first type of subleading terms, which have the same scaling behavior as the leading term (3.47) with respect to the charges $q_i, r$. Corrections of the second type are linear in the charges, hence they have a lower scaling behavior.

The microscopic formula becomes

$$S_{\text{micro}} \sim \pi \sqrt{2rn\left(Q - cr + \frac{dr^2}{3}\right) - r^2 m^2}. \tag{3.51}$$

The question is if the subleading terms present in (3.51) can be understood from a supergravity analysis.

Let us first try to understand the origin of such corrections in F-theory. So far we have been working with tree level $N = 1$ supergravity, which can be regarded as a truncation of the $N = 4$ theory. However the low energy description of F-theory has extra couplings which are not consistent with a truncation of the $N = 4$ theory. The couplings in question are six-dimensional Green-Schwartz terms required by anomaly cancellation [84–89]. In this paper we consider only F-theory compactifications on smooth elliptic fibrations, therefore we do not have six dimensional vector multiplets. The theory will have only gravitational anomalies, which determine the higher curvature corrections to the tree level supergravity action.

According to [84, 87–89] the higher curvature terms are encoded in a shift of the elementary field strengths $H^i = dB^i$ by a gravitational Chern-Simons term. More precisely, one has to define

$$H^i = dB^i + a_i\omega \tag{3.52}$$

where $\omega$ is the gravitational Chern-Simons term for the six dimensional spin connection. According to [88], the coefficients $a_i$ are given by $a_i = \frac{c_i}{2}$, where the $c_i$ were defined in (2.23). The effect of this shift on the supersymmetry variations and equations of motion has been worked out in [84, 87, 89]. In principle one should solve the new equations of motion and BPS conditions in order to understand the effect of higher curvature corrections on the black string entropy. This would be quite an involved analysis which we will leave for future work.

However, let us observe that if we ignore the back-reaction of the noncritical string on the six dimensional space-time geometry, the shift (3.52) results in a shift of the form

$$q^i \rightarrow q^i - \frac{rc_i}{2} \tag{3.53}$$

on the charges. This follows by a direct evaluation of the Chern-Simons term in a Taub-NUT background. Such a shift is reminiscent of a similar modification of black hole charges in the four dimensional attractor mechanism [58]. In fact it can be easily checked that this is indeed the shift predicted in [58] for the attractor solutions discussed in section 4.1. One can think about the correction term in (3.52) as giving rise to a difference between the charge measured at infinity, $q^i$, and the actual charge of the black hole, $q^i - \frac{rc_i}{2}$. Accepting this shift on a conjectural basis for the moment, note that it would result in a modified macroscopic entropy formula of the form

$$ S_{\mathrm{macro}} = \pi \sqrt{2rn\left(Q - cr + \frac{dr^2}{4}\right) - r^2 m^2}. \tag{3.54} $$

Quite remarkably, this formula exhibits the same subleading correction as the microscopic result (3.50), but the next order corrections, namely the terms proportional to $nr^3d$, are different. These terms are very small in the limit (3.46), but they would become important in a regime in which $q_i$ and $rc_i$ are of the same order of magnitude. This is precisely the regime in which we also expect the effects of the singular divisors on the microscopic entropy formula to be become important, as explained below (3.16). It would be very interesting to confirm the macroscopic entropy formula (3.54) by a direct supergravity computation. If the result conjectured here is indeed valid, it would also be very interesting to understand the microscopic computation in the regime $q_i \sim rc_i$ and compare the two expressions.

## A. Charges and Fourier-Mukai transform

In this appendix we rewrite the D-brane charges obtained by Fourier-Mukai transform in terms of a natural basis of periods from the point of view of homological mirror symmetry. This will make the integrality of charges manifest. Although not crucial for the considerations of this paper, this is still an important consistency check for the formalism.

In order to obtain an expression with integral coefficients we have to expand $Z(\mathcal{F})$ in terms of a basis of periods consisting for central charges of D-branes on $X$. More precisely, homological mirror symmetry implies that there should exist a collection of bundles (or more generally derived objects) $E_\Lambda, F^\Lambda$ on $X$, $\Lambda = 1, \ldots, h^{1,1}(X) + 1$, so that the central charges $Z(E_\Lambda), Z(F^\Lambda)$ form a symplectic basis of periods near the large radius limit point. In particular we should have the following D-brane intersection matrix

$$ \int_X \mathrm{ch}(E_\Lambda) \wedge \mathrm{ch}((F^{\Lambda'})^\vee) \wedge \mathrm{Td}(X) = \epsilon_\Lambda^{\Lambda'} \tag{A.1} $$

where $F^\vee$ denotes the (derived) dual of $F$ and $\epsilon_\Lambda^{\Lambda'}$ is the canonical antisymmetric tensor, with all the other entries in the intersection matrix vanishing. Note that it suffices to know the K-theory classes of $E_\Lambda, F^\Lambda$. In order to construct such a basis in our class of examples, let $F^A$ be a collection of torsion sheaves on $X$ so that $\mathrm{ch}(F^A) = \beta^A$, $A = 1, \ldots, h^{1,1}(X)$. Take

$$ E_A = \mathcal{O}_X(\alpha_A) - \mathcal{O}_X - \mathcal{O}_p^{\oplus n_A}, $$

$A = 1, \ldots, h^{1,1}(X)$, in K-theory, where $\mathcal{O}_X(D)$ denotes the line bundle on $X$ with first Chern class $D$, $\mathcal{O}_p$ denotes the structure sheaf of a point $p \in X$, and

$$n_A = \int_X \alpha_A \wedge \mathrm{Td}_2(X).$$

Moreover let $E_0 = \mathcal{O}_X$, $F^0 = \mathcal{O}_p$. Then, using the intersection numbers (2.22), it is straightforward to check that the intersection relations (A.1) hold if and only if

$$\int_B (\gamma_j \wedge \gamma_j - \gamma_j \wedge c_1(B)) = 0 \tag{A.2}$$

for all $j = 1, \ldots, h^{1,1}(B)$. We will assume this relation to be satisfied from now on. Then a direct computation using the intersection relations (A.2) and

$$\alpha_h^2 = (\sigma + \pi^* c_1(B))^2 = c_i \beta^i + d\beta^h, \qquad \alpha_i^2 = (\gamma_j)_B^2 \beta^h \tag{A.3}$$

yields

$$\begin{aligned} Z(\mathcal{F}) = & r(Z(E_h)) + (q_i - rc_i)d^{ij}(Z(E_j)) + mZ(F^h) - rc_i Z(F^i) \\ & - \left(n + n_h + (q_i - rc_i)d^{ij}n_j\right) Z(F^0). \end{aligned} \tag{A.4}$$

Now recall that the basis $B$ of a smooth elliptic fibration with a section must be a del Pezzo surface, which has a unimodular middle cohomology lattice. Then it follows that all coefficients in this expansion are integral, as expected. Moreover, all terms in this expressions are covariant with respect to any linear change of basis $\{\gamma_j\}$ which preserves the intersection relations (A.2). This expected, since such transformations are mapped to symplectic changes of basis by mirror symmetry.

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